PMM U.S.S.R.,Vol.48,No.2,pp.217- 223,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

ON CERTAIN CONTACT PROBLEMS FOR REINFORCED PLATES*

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Problems of the smooth contact with the linearly deformable base of an infinite plate reinforced by an thin elastic rib/stiffener/(of finite or semi-infinite length) are examined. The rib is subjected to a normal load while its ends are free. The bending moments in the reinforcing ribs are computed. Analogous problems are ecamined in /1/ for reinforcement by infinite ribs. The characteristic feature of the investigation is the reliance on solutions with non-integrable singularities for the integral equations of the contact problems mentioned, and the utilization of integrals in a regularized (generalized) sense /2/. The beginning of similar investigations is found in /3—5/.

1. Consider the problem of the bending of an infinite plate $-\infty < x,\ y < \infty$, lying without friction on a linearly deformable base /6/. The plate is reinforced along the line $x = 0, |y| < a$ by a thin elastic rib to which a load $q(y)$, directed perpendicularly to the (x, y) plane, is applied. It is required to find the bending moment in the rib and the contact interaction forces between the rib and the plate.

The presence of the reinforcing rib causes a jump in the generalized transverse force V_x in the plate. Using the notation $\langle f\rangle = f(-0,\ y) - f(+0, y)$, we have

$$\langle w\rangle = \langle w_x'\rangle = \langle M_x\rangle = 0,\ \ \langle V_x\rangle = \varphi(y) \tag{1.1}$$

Here $\varphi(y)$ is the unknown contact interaction force between the rib and the plate, where $\varphi(y) \equiv 0$ for $|y| > a$.

Considering the ends of the rib to be free, we arrive at the following boundary value problem for the rib deflection $v(y)$:

$$E_* I_* v^{(4)}(y) = q(y) - \varphi(y)\ (|y| < a),\ \ v''(\pm a) = v'''(\pm a) = 0 \tag{1.2}$$

where E_* is the modulus of elasticity of the rib material, and I_* is the rib moment of inertia.

Let $p(x, y)$ be the contact force between the plate and the base. Then the plate deflection $w(x, y)$ should satisfy the differential equation

$$D\Delta^2 w = -p(x, y),\ D = Eh^3[12(1-\nu^2)]^{-1} \tag{1.3}$$

for all $x \in (-\infty, \infty), y \in (-\infty, \infty)$, except $x = 0, |y| < a$ (D is the plate stiffness, h is its thickness, E is the modulus of elasticity, and ν is Poisson's ratio).

The vertical displacement of the surface points of the base $w_0(x,\ y)$ can be fonnd from the formula

$$w_0(x, y) = \iint_{-\infty}^{\infty} K(x-\xi, y-\eta)\, p(\xi, \eta)\, d\xi\, d\eta \tag{1.4}$$

where $K(x, y)$ is the kernel of the linearly deformable base /6/ for which the following representation holds

$$K(x, y) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} H(\alpha, \beta)\, e^{-i(\alpha x + \beta y)}\, d\alpha\, d\beta$$

With respect to the density of the base kernel $H(\alpha, \beta)$ we assume that it has the asymptotic form

$$\begin{aligned} H(\alpha, \beta) &= O(\alpha^{\varepsilon_1 - 1}),\ |\alpha| \to \infty,\ 0 \leqslant \varepsilon_1 < 1 \\ H(\alpha, \beta) &= O(\beta^{\varepsilon_2 - 1}),\ |\beta| \to \infty,\ 0 \leqslant \varepsilon_2 < 1 \end{aligned} \tag{1.5}$$

which is satisfied for the majority of known bases. In particular, for an elastic isotropic half-space $H(\alpha, \beta) = \theta(\alpha^2 + \beta^2)^{-1/2}$, $\theta = 2(1-\nu_0^2)E_0^{-1}$, where E_0 and ν_0 are the modulus of elasticity and Poisson's ratio of the half-space material.

We will reduce problem (1.2)—(1.4) to an integral equation in $\varphi(y)$. To do this, taking (1.1) into account we apply a Fourier transformation in x by the scheme of the generalized

*Prikl.Matem.Mekhan.,48,2,307-314,1984

method /5/ and a Fourier transformation in y to (1.3 and (1.4). Assuming the contact to be bilateral, i.e., $w(x, y) = w_0(x, y)$, by eliminating the transform of the contact forces $p(x, y)$ and inverting the Fourier transforms, we obtain

$$w(x, y) = \int_{-a}^{a} R(x, y-\xi)\varphi(\xi)\,d\xi \tag{1.6}$$

$$R(x, y) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \frac{e^{-i(\alpha x+\beta y)}}{\Psi(\alpha, \beta)}\,d\alpha\,d\beta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\beta(x)\,e^{-i\beta y}\,d\beta$$

$$\Phi_\beta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{\Psi(\alpha, \beta)}\,d\alpha, \quad \Psi(\alpha, \beta) = \frac{1}{H(\alpha, \beta)} + D(\alpha^2+\beta^2)^2$$

We find the rib deflection from the formula

$$v(y) = \frac{1}{E_* I_*}\int_{-a}^{a} G(y, \xi)[q(\xi) - \varphi(\xi)]\,d\xi + V_0 + V_1 y \tag{1.7}$$

where V_0, V_1 are constants determined later, and $G(y, \xi)$ is the generalized Green's function /7/ of the boundary value problem (1.2) which has the following form for the interval $(-a, a)$:

$$G(y, \xi) = \frac{1}{12}|y-\xi|^3 - \frac{1}{80a^3}(y^4+\xi^4)y\xi - \frac{1}{48a}(y^4+\xi^4) +$$
$$\frac{1}{8}(y^2+\xi^2)y\xi - \frac{a}{8}(y^2+\xi^2) + \frac{33a}{280}y\xi + \frac{a^3}{40}$$

Formula (1.7) holds under the following conditions on the right side of the differential equation from (1.2)

$$\int_{-a}^{a} [q(y) - \varphi(y)]\,dy = 0, \quad \int_{-a}^{a} [q(y) - \varphi(y)]\,y\,dy = 0 \tag{1.8}$$

which is satisfied because of the selfequilibration of the load applied to the rib with free ends.

By realizing the condition of rib and plate contact $v(y) = w(0, y)$ and taking account of (1.6) and (1.7), we obtain an integral equation in $\varphi(y)$

$$\int_{-a}^{a} l(y, \xi)\varphi(\xi)\,d\xi = V_0 + V_1 y + \frac{1}{E_* I_*}\int_{-a}^{a} G(y, \xi)q(\xi)\,d\xi \tag{1.9}$$

$$(|y| < a), \quad l(y, \xi) = R(0, y-\xi) + \frac{1}{E_* I_*}G(y, \xi)$$

Let us clarify the form of the singularities of the desired function $\varphi(\xi)$ at the points $\xi = \pm a$ for which we extract the principal part in the kernel $l(y, \xi)$. The differential properties of the function $R(0, z)$ in the neighbourhood of the point $z = 0$ are determined by the behaviour of the functions $\Psi(\alpha, \beta)$ as $\alpha \to \infty$ and $\beta \to \infty$. Taking account of (1.5) and (1.6), we obtain $\Psi(\alpha, \beta) = O((\alpha^2+\beta^2)^2)$. Using the integral

$$R_0(0, z) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \frac{e^{-i\beta z}}{(\alpha^2+\beta^2)^2}\,d\alpha\,d\beta = \frac{z^2}{8\pi}\left(\ln|z| - \frac{3}{2} + \gamma\right)$$

(γ is Euler's constant) which is obtained by using the appropriate formulas from /8, 9 p.104/, we arrive at the conclusion that

$$l(y, \xi) = \frac{1}{8\pi D}(y-\xi)^2\ln|y-\xi| + l_0(y, \xi)$$

where $l_0(y, \xi)$ is a smoother function than the first term.

Equation (1.9) obtained is an integral equation of the first kind with a continuous kernel. Following /4, 5/, we seek its solution in the class of functions with non-integrable singularities at the ends of the interval

$$\varphi(\xi) = (a^2 - \xi^2)^{-3/2}\varphi_0(\xi) \tag{1.10}$$

($\varphi_0(\xi)$ satisfies the Hölder condition), and the corresponding integrals are understood in the regularized sense /2/.

Note that utilization of the regularization of divergent integrals enables one, in principle, to solve (1.9), and in the broader classes of functions

$$\varphi(\xi) = (a-\xi)^{\omega_1}(a+\xi)^{\omega_2}\varphi_0(\xi) \quad (\operatorname{Re}\omega_1 \leqslant -2, \operatorname{Re}\omega_2 \leqslant -2) \tag{1.11}$$

but, in these classes (1.9) has infinitely many solutions. In particular, the homogeneous characteristic equation

$$\int_{-a}^{a} (y-\xi)^2 \ln|y-\xi|\,\varphi(\xi)\,d\xi = 0$$

has infinitely many solutions of the form

$$\varphi_{mn}(\xi) = (a-\xi)^{-3/2-m}(a+\xi)^{-3/2-n} \quad (m = 0, 1, 2, \ldots, n = 0, 1, 2, \ldots, m+n \geqslant 1)$$

and only the trivial solution $\varphi(\xi) \equiv 0$ in the class (1.10).

It should be noted that for the formulation of the problem in question to be correct, the following fact is very important. The class of solutions (1.11) is characterized by the fact that the energy integral of a bent plate

$$\frac{Eh^3}{12(1+\nu)}\iint_S \left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w^2}{\partial y^2}\right)^2 + 2(1-\nu)\left(\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)\right] dS$$

diverges while the second derivatives of $w(x, y)$ for selecting the contact forces in the class (1.10) behave as $r^{-1/2}$ on approaching the points $x = 0, y = \pm a$, and the energy integral is convergent as an improper one, which enables the question of the uniqueness of the solution to be investigated by known methods.

The approximate solution of (1.9) is obtained by the method of orthogonal polynomials /5, 6/. We consider the case of an even external load $q(y)$. Following /4, 5/ and going over to the interval $(-1,1)$ by using the substitution $y = at$, $\xi = a\tau$ we write the desired function in the form

$$\varphi(a\tau) = \psi(\tau) = \sum_{j=0}^{\infty} \psi_j \pi_j(\tau), \quad \pi_0(\tau) = (1-\tau^2)^{-1/2} \tag{1.12}$$

$$\pi_j(\tau) = (1-\tau^2)^{-3/2}\, 2\sqrt{\pi}\,(2j)!\,[\Gamma(2j-{}^1/_2)]^{-1} P_{2j}^{-3/2,\,-3/2}(\tau) \quad (j \geqslant 1)$$

Substituting (1.12) into (1.9), multiplying by $Da^{-3}\pi_i(t)$ $(i = 0, 1, 2, \ldots)$ and integrating between -1 and 1 with respect to t, we arrive at the final system of linear algebraic equations

$$\sum_{j=0}^{\infty} a_{ij}\psi_j = b_i \quad (i = 0, 1, 2, \ldots) \tag{1.13}$$

$$a_{ij} = \int_0^\infty \frac{D}{\pi a^2}\Phi_\beta(0) A_i(a\beta) A_j(a\beta)\,d\beta + c\sum_{n=1}^{\infty}\frac{A_i(\pi n) A_j(\pi n)}{\pi^4 n^4} +$$

$$2c\left(\sum_{m=1}^{\infty}\frac{(-1)^m A_i(\pi m)}{\pi^2 m^2}\right)\left(\sum_{n=1}^{\infty}\frac{(-1)^n A_j(\pi n)}{\pi^2 n^2}\right), \quad b_i = \pi V_0 D a^{-3}\delta_{i0} +$$

$$c\int_{-1}^{1}\left[\sum_{n=1}^{\infty}\frac{A_i(\pi n)\cos\pi n\tau}{\pi^4 n^4} + \frac{1}{2}\left(\tau^2-\frac{1}{3}\right)B_i\right] q(a\tau)\,d\tau,$$

$$c = \frac{aD}{E_* I_*}, \quad A_0(b) = \pi J_0(b).$$

$$A_i(b) = (-1)^i \pi b J_{2i-1}(b) \quad (i \geqslant 1), \quad B_0 = \frac{\pi^2}{24},$$

$$B_1 = -\frac{\pi}{4}, \quad B_i = 0 \quad (i \geqslant 2)$$

Expressions for a_{ij}, b_i are obtained by expanding the function $G(y, \xi)$ in the trigonometric series

$$G(at, a\tau) = a^3\sum_{n=1}^{\infty}\left\{\left[\frac{\cos\pi nt}{\pi^4 n^4} + \frac{(-1)^n}{2\pi^2 n^2}\left(t^2-\frac{1}{3}\right)\right]\cos\pi n\tau + \right.$$

$$\left.\left[\frac{\sin\pi nt}{\pi^4 n^4} + \frac{(-1)^n}{\pi n}\left(\frac{1}{40}t^5 - \frac{1}{12}t^3 + \left(\frac{11}{280}+\frac{3}{\pi^4 n^4}\right)t\right)\right]\sin\pi n\tau\right\} =$$

$$\sum_{n=1}^{\infty}\frac{\cos\pi nt\cos\pi n\tau}{\pi^4 n^4} + 2\left(\sum_{m=1}^{\infty}\frac{(-1)^m\cos\pi mt}{\pi^2 m^2}\right)\left(\sum_{n=1}^{\infty}\frac{(-1)^n\cos\pi n\tau}{\pi^2 n^2}\right) +$$

$$\sum_{n=1}^{\infty}\frac{\sin\pi nt\sin\pi n\tau}{\pi^4 n^4} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{4}{105} - \frac{6}{\pi^4 n^4} - \frac{6}{\pi^4 m^4}\right)\frac{\sin\pi mt\sin\pi n\tau}{(-1)^{m+n}\pi^2 mn}$$

An unknown constant V_0, equal to the deflection of the ribs as a rigid whole, is present on the right side of (1.13). Consequently, we find the coefficient ψ_0 by substituting (1.12) into (1.8). The second condition is automatically satisfied here because of the evenness of $q(y)$ and $\varphi(y)$; the former yields

$$\psi_0 = \frac{1}{\pi}\int_{-1}^{1} q(a\tau)\,d\tau$$

Furthermore $\psi_j\ (j \geqslant 1)$ are found from (1.13) for $i \geqslant 1$, and V_0 from (1.13) for $i = 0$.

System (1.13) is quasiregular, and its approximate solution can be obtained by the method of reduction /10/.

To prove the quasiregularity of system (1.13), we examine a_{ij} for $i \geqslant 1,\ j \geqslant 1$. According to (1.5) and (1.6), it is possible to write $D\Phi_\beta(0) = {}^1/_4\,|\beta|^{-3} + G_0(\beta)$, where $G_0(\beta) = O(\beta^{-6-\varepsilon})$ as $\beta \to \infty$. Then

$$a_{ij} = \frac{\pi\delta_{ij}}{8(2i-1)} + (-1)^{i+j} a_{ij,\,1} + (-1)^{i+j} c a_{ij,\,2} + c\frac{\pi^2}{8}\delta_{i1}\delta_{j1} \tag{1.14}$$

$$a_{ij,\,1} = \pi\int_0^\infty \beta^2 G_0(\beta)\, J_{2i-1}(a\beta)\, J_{2j-1}(a\beta)\,d\beta = \int_0^\pi\int_0^\pi \sin(2i-1)\theta\,\sin(2j-1)\varphi\times \tag{1.15}$$

$$\left(\frac{1}{\pi}\int_0^\infty \beta^2 G_0(\beta)\sin(a\beta\sin\theta)\sin(a\beta\sin\varphi)\,d\beta\right)d\theta\,d\varphi$$

$$a_{ij,\,2} = \sum_{n=1}^{\infty}\frac{J_{2i-1}(\pi n)\,J_{2j-1}(\pi n)}{n^2} = \frac{(-1)^{i+j}\,[{}^1/_4 - (i-j)^2]^{-1}}{4(i+j-{}^3/_2)(i+j-{}^1/_2)} \quad (i+j > 2) \tag{1.16}$$

The Kronecker delta δ_{ij} is obtained from 6.538(2) in /8/, the second equality in (1.15) is obtained from the integral representation of the Bessel function, and the series (1.16) is summed in /6, p.158/. It follows from the asymptotic form of $G_0(\beta)$ that $|a_{ij,\,1}| < M(2i-1)^{-2}(2j-1)^{-2}$, where M is independent of i and j. Then (1.13) can be rewritten for $i \geqslant 2$ in the form

$$\psi_i + \sum_{j=0}^{\infty} c_{ij}\psi_j = f_i$$

where the following estimate holds for c_{ij}:

$$|c_{ij}| < \frac{8(2i-1)}{\pi}\left[\frac{c\,|{}^1/_4 - (i-j)^2|^{-1}}{4(i+j-{}^3/_2)^2} + \frac{M}{(2i-1)^2(2j-1)^2}\right]$$

and $\sigma_i = |c_{i0}| + |c_{i1}| + \ldots < 1$ starting with a certain i and fairly small c. Substituting (1.12) into (1.7), we obtain an expresion for the rib deflection in the form

$$v(at) = \frac{1}{E_* I_*}\int_{-a}^{a} G(y,\xi)\,q(\xi)\,d\xi +$$

$$\frac{a^4}{E_* I_*}\left(\sum_{j=0}^{\infty}\psi_j\sum_{n=1}^{\infty}\frac{\cos\pi nt\,A_j(\pi n)}{\pi^4 n^4} + \frac{\pi}{48}\left(t^2 - \frac{1}{3}\right)(\psi_0 + 6\psi_1)\right) + V_0$$

The expression for the bending moment is obtained in the same way as (6.1.58) in /5/ and when $q(y) = P\delta(y)$ has the form

$$M(at) = Pa\left(-\frac{|t|}{2} + \frac{t\arcsin t + \sqrt{1-t^2}}{\pi} + \frac{\sqrt{1-t^2}}{\pi}\sum_{j=1}^{\infty}\psi_j\frac{U_{2j-2}(t)}{(2j-1)}\frac{a}{P}\right)$$

The results of calculations of the quantity $10^3(Pa)^{-1}M(at)$ are represented in Table 1 as a function of the quantity $\lambda = ab^{-1}$, the ratio of the rib length to its breadth. An elastic half-space is taken as the foundation and the following relations are used for the elastic parameters of the plate, the rib, and the foundation: $\nu = \nu_0 = {}^1/_3$, $E/E_* = {}^1/_2$, $E/E_0 = 10$, $h/H = {}^1/_2$ (H is the rib height). Extraction of the non-integrable singularities in the solution of the equation assured rapid convergence of series (1.12), and retention of six terms is sufficient to obtain three significant figures.

2. We consider the problem of the bending of an unbounded plate, reinforced by a semi-infinite rib $x = 0,\ y \geqslant 0$ whose end $y = 0$ is simply supported, on a linearly deformable foundation. Equation (1.3) holds everywhere except at the points $x = 0,\ y \geqslant 0$, and we have instead of (1.2)

$$E_* I_* v^{(4)}(y) = q(y) - \varphi(y),\quad v''(0) = v'''(0) = 0 \tag{2.1}$$

where the unknown function $\varphi(y)$ equals zero for $y < 0$.

In place of (1.6) we have the following for the plate deflection:

$$w(x, y) = \int_0^\infty R(x, y-\xi)\varphi(\xi)\,d\xi \tag{2.2}$$

Taking account of the equation $v(y) = w(0, y)$, we substitute (2.2) for $x = 0$ into (2.1). We consequently arrive at a Wiener-Hopf type integro-differential equation in the function $\varphi(y)$

$$\varphi(y) + b\frac{d^4}{dy^4}\int_0^\infty l_1(y-t)\varphi(t)\,dt = q(y) \quad (0 \leqslant y < \infty) \tag{2.3}$$

$$l_1(z) = 2\pi R(0, z) = \int_{-\infty}^{\infty} \Phi_\beta(0)e^{-i\beta z}\,d\beta, \quad b = \frac{E_* I_*}{2\pi}$$

The integro-differential equation (2.3) allows of exact solution by the method of factorization, but the corresponding formulas are inconvenient for numerical realization. According to /6/, the known factorization formulas are here transformed to a form convenient for calculations.

Table 1

t	$\lambda = 2$	4	10	20
0.0	495	400	345	301
0.1	446	354	303	264
0.2	404	316	275	249
0.3	364	283	256	248
0.4	326	252	240	247
0.5	288	223	224	246
0.6	250	195	207	244
0.7	211	166	189	238
0.8	169	135	163	214
0.9	118	96	123	170
1.0	0	0	0	0

Table 2

y	$\varkappa = 1.5$	2,0	2,5
0.1	310	185	121
0.2	339	201	133
0.4	353	200	131
0.6	349	186	121
0.8	342	171	108
1.0	334	155	97
1.5	317	123	71
2.0	304	100	53
3.0	288	73	32
4.0	281	61	23
5.0	278	56	19

The solution of (2.3) has the form

$$\varphi(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(-\zeta)\varphi_\zeta(y)\,d\zeta, \quad Q(\zeta) = \int_{-\infty}^{\infty} q(y)e^{i\zeta y}\,dy \tag{2.4}$$

$$\varphi_\zeta(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[B(u) + i\frac{L_1^-(-\zeta)}{u+\zeta}\right]L_1^+(u)e^{-iuy}\,du$$

where $L_1^\pm(u)$ is the solution of the factorization problem

$$[1 + bu^4L_1(u)]^{-1} = L_1^+(u)L_1^-(u), \quad L_1(u) = \int_{-\infty}^{\infty} l_1(z)e^{iuz}\,dz \tag{2.5}$$

and $B(u)$ is an integral function. It is ordinarily determined by starting from the conditions for convergence of the integral

$$\int_{-\infty}^{\infty} B(u)L_1^+(u)e^{-iuy}\,du \tag{2.6}$$

Since $L_1(u) = 2\pi\Phi_u(0)$, then $L_1^+(u) = O(u^{-1/2})$, $u \to \infty$, and the function $B(u)$ should be a constant. But, on the other hand, satisfaction of the two conditions from (2.1) at the end of the rib $y = 0$ is required, whereupon it is necessary to have two arbitrary constants A_0, A_1, i.e, we should take

$$B(u) = \sum_{j=0}^{1} A_j(u+i) \tag{2.7}$$

If integral (2.6) is understood here in the regularized sense of /2/, then on the basis of (2.4) and the well-known formula /9, p.103/, we find that the solution $\varphi(y) = O(y^{-3/2})$ as $y \to 0$. Therefore, as in the previous problem we arrive at a solution with non-integrable singularities. Utilization of the Fourier transformation of generalized functions /9/ even in this case enables us to take a polynomial of any power n as $B(u)$, but for $n \geqslant 2$ the solution will not be unique, and the energy integral will be divergent.

Transforming (2.4) to a form convenient for calculations by the scheme mentioned in /6/, we obtain a solution of (2.3) in the form

$$\varphi(y)=\frac{1}{\lambda}e^{-y}\Big[\sum_{j=0}^{1}B_jy^{j-3/2}\sum_{m=0}^{\infty}g_m\mu_{mj}L_m^{j-3/2}(2y)+ \qquad (2.8)$$

$$y^{-3/2}\sum_{m=0}^{\infty}\varphi_m\mu_{m0}L_m^{-1/2}(2y)\Big],\quad \mu_{mj}=m!\left[\Gamma\left(m+j-\frac{3}{2}\right)\right]^{-1}$$

$$\varphi_m=\frac{i}{4\pi}\sum_{k=0}^{m}\sum_{s=0}^{k}g_{k-s}\frac{(-2)_s}{s!}\int_{-\infty}^{\infty}Q(-\zeta)\frac{L_1^-(-\zeta)}{\zeta+i}\left(\frac{\zeta-i}{\zeta+i}\right)^{m-k}d\zeta$$

$$g_0=e^{h_0},\quad g_n=\sum_{m=0}^{n-1}\frac{n-m}{m}h_{n-m}g_m\quad (n=1,2,\ldots)$$

$$h_0=\frac{1}{2\pi}\int_0^{\pi}\ln\left[R\left(\mathrm{tg}\,\frac{\theta}{2}\right)\right]d\theta,$$

$$h_n=\frac{(-1)^n}{\pi}\int_0^{\pi}\ln\left[R\left(\mathrm{tg}\,\frac{\theta}{2}\right)\right]\cos n\theta\, d\theta$$

$$R(u)=\lambda^2\sqrt{1+u^2}\,[1+bu^4L_1(u)]^{-1},\quad \lambda^2={}^1/_2\pi bD^{-1}$$

Here B_0 and B_1 are new arbitrary constants. We determine them from the rib equilibrium conditions which are equivalent to the boundary conditions in (2.1)

$$\int_0^{\infty}[q(y)-\varphi(y)]\,dy=0,\quad \int_0^{\infty}y\,[q(y)-\varphi(y)]\,dy=0 \qquad (2.9)$$

If the external load is a concentrated force P applied to the rib at the point $y=d$: $q(y)=P\delta(y-d)$, then

$$\varphi_m=-\frac{P\sqrt{d}}{2\lambda}e^{-d}\sum_{k=0}^{m}\sum_{s=0}^{k}g_{k-s}\frac{(-2)_s}{s!}\sum_{n=0}^{\infty}g_n\mu_{m+n-k,\,2}L_{m+n-k}^{1/2}(2d)$$

If the force is applied to the point $y=0$, then $\varphi_m=0$,(2.9) and yields the following conditions for finding B_0 and B_1:

$$\sum_{j=0}^{1}B_j\sum_{m=0}^{\infty}(-1)^m g_m=-\lambda P,$$

$$\sum_{j=0}^{1}B_j\sum_{m=0}^{\infty}(-1)^m g_m\left(2m+j-\frac{1}{2}\right)=0$$

To determine the bending moment in the rib, we integrate the differential equation from (2.1) twice, finding the constants of integration by using the boundary conditions from (2.1). We consequently obtain

$$M_p(y)=-\frac{1}{\lambda}\sum_{j=0}^{1}B_j\sum_{m=0}^{\infty}g_m\mu_{mj}J_m^{j-3/2}(y)$$

$$J_0^{\alpha}(y)=-\Gamma(\alpha+1)\,y+\sum_{n=0}^{\infty}\frac{(-1)^n y^{n+\alpha+2}}{n!\,(n+\alpha+1)(n+\alpha+2)}$$

$$J_1^{\alpha}(y)=-\Gamma(\alpha+2)\,y+\sum_{n=0}\frac{(-1)^n(2n+3\alpha+3)\,y^{n+\alpha+2}}{n!\,(n+\alpha+1)(n+\alpha+2)}$$

$$J_m^{\alpha}(y)=\frac{2}{m(m-1)}e^{-y}y^{\alpha+2}L_{m-2}^{\alpha+2}(y)+\frac{m+\alpha}{m}J_{m-1}^{\alpha}(y)\quad (m\geqslant 2)$$

The results of calculations of the quantity $10^3P^{-1}M_p(y)$ are given in Table 2 as a function of the quantity $\varkappa=H/h$ for the same elastic parameters as in the first problem.

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Translated by M.D.F.

PMM U.S.S.R., Vol.48, No.2, pp.223-230, 1984
Printed in Great Britain
0021-8928/84 $10.00+0.00

A NON-STATIONARY DYNAMICAL PERIODIC CONTACT PROBLEM FOR A HOMOGENEOUS ELASTIC HALF-PLANE*

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The problem of determining the contact stresses under a periodic system of stamps located on the boundary of a homogeneous elastic half-plane and moving under the effect of a load, identical for all stamps, that is arbitrary in time, is investigated. The problem reduces to solving a Fredholm integral equation of the first kind for the Laplace transform of the contact stresses. The stresses are sought in the form of a double expansion in Chebyshev polynomials of the linear coordinate and Laguerre polynomials of time. The coefficients of the expansions are determined recursively from an infinite quasiregular system of linear algebraic equations.

Despite the fact that the static periodic contact problems of the theory of elasticity, on the one hand (/1-6/, say), and dynamic problems for a finite number if stamps on the other (see the survey in /17/), have been studied repeatedly by different investigators, so far as we know, the plane non-stationary dynamical periodic contact problem has still not been examined at all.

1. A system of vertical unit impulses at the points

$$x = ml\ (m = 0, \pm 1, \pm 2, \ldots) \quad p(x, t) = \sum_{m=-\infty}^{\infty} \delta(x - ml)\,\delta(t) \tag{1.1}$$

where ($\delta(t)$ is the delta function), is applied to the boundary of a homogeneous elastic half-plane. The Ox axis is directed along the half-plane boundary. The variable x and the time t are assumed to be dimensionless; the length scale is a and the time scale is a/c_2. Here a is a certain parameter with the dimensions of length, and c_2 is the transverse velocity of wave propagation in an elastic half-space.

Substituting (1.1) into (1.24) in /7/ and using the equation

$$\sum_{m=-\infty}^{\infty} e^{iml\xi} = \frac{2\pi}{l} \sum_{k=-\infty}^{\infty} \delta\left(\xi - k\frac{2\pi}{l}\right)$$

we obtain a function $\bar{v}_1(x, s)$ that is the Laplace transform of the vertical displacement of a boundary point of the half-plane with abscissa x due to the action of a periodic system of concentrated unit impulses

*Prikl.Matem.Mekhan., 48, 2, 315-323, 1984